Two parameter Deformed Multimode Oscillators and q-Symmetric States

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Abstract

Two types of the coherent states for two parameter deformed multimode oscillator system are investigated. Moreover, two parameter deformed $gl(n)$ algebra and deformed symmetric states are constructed.

1 Introduction

Quantum groups or q-deformed Lie algebra implies some specific deformations of classical Lie algebras.

From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6-10].

Recently Greenberg [11] has studied the following q-deformation of multi mode boson algebra:

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij},$$

where the deformation parameter q has to be real. The main problem of Greenberg's approach is that we can not derive the relation among a_i's operators at all. In order to resolve this problem, Mishra and Rajasekaran [12] generalized the algebra to complex parameter q with $|q| = 1$ and another real deformation parameter p. In this paper we use the result of ref [12] to construct two types of coherent states and q-symmetric ststes.

2 Two Parameter Deformed Multimode Oscillators

2.1 Representation and Coherent States

In this subsection we discuss the algebra given in ref [12] and develop its reprsentation. Mishra and Rajasekaran's algebra for multi mode oscillators is given by

$$a_i a_j^\dagger = q a_j^\dagger a_i \quad (i < j)$$

$$a_i a_i^\dagger - p a_i^\dagger a_i = 1$$

$$a_i a_j = q^{-1} a_j a_i \quad (i < j), \tag{1}$$

where $i, j = 1, 2, \cdots, n$. In this case we can say that $a_i^\dagger$ is a hermitian adjoint of a_i.

The fock space representation of the algebra (1) can be easily constructed by introducing the hermitian number operators $\{N_1, N_2, \cdots, N_n\}$ obeying

$$[N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \quad (i, j = 1, 2, \cdots, n). \tag{2}$$

From the second relation of eq.(1) and eq.(2), the relation between the number operator and creation and annihilation operator is given by

$$a_i^\dagger a_i = [N_i] = \frac{p^{N_i} - 1}{p - 1} \tag{3}$$

or

$$N_i = \sum_{k=1}^{\infty} \frac{(1-p)^k}{1-p^k} (a_i^\dagger)^k a_i^k. \tag{4}$$

Let $|0, 0, \cdots, 0>$ be the unique ground state of this system satisfying

$$N_i|0, 0, \cdots, 0>= 0, \quad a_i|0, 0, \cdots, 0>= 0, \quad (i, j = 1, 2, \cdots, n) \tag{5}$$

and $\{|n_1, n_2, \cdots, n_n > |n_i = 0, 1, 2, \cdots\}$ be the complete set of the orthonormal number eigenstates obeying

$$N_i|n_1, n_2, \cdots, n_n >= n_i|n_1, n_2, \cdots, n_n > \tag{6}$$

and

$$< n_1, \cdots, n_n|n_1', \cdots, n_n' >= \delta_{n_1 n_1'} \cdots \delta_{n_2 n_2'}. \tag{7}$$

If we set

$$a_i|n_1, n_2, \cdots, n_n >= f_i(n_1, \cdots, n_n)|n_1, \cdots, n_i - 1. \cdots, n_n >, \tag{8}$$

we have, from the fact that $a_i^\dagger$ is a hermitian adjoint of a_i,

$$a_i^\dagger|n_1, n_2, \cdots, n_n >= f^*(n_1, \cdots, n_i + 1, \cdots, n_n)|n_1, \cdots, n_i + 1. \cdots, n_n > . \tag{9}$$

Making use of relation $a_i a_{i+1} = q^{-1} a_{i+1} a_i$ we find the following relation for f_i's:

$$q \frac{f_{i+1}(n_1, \cdots, n_n)}{f_{i+1}(n_1, \cdots, n_i - 1, \cdots, n_n} = \frac{f_i(n_1, \cdots, n_n)}{f_i(n_1, \cdots, n_{i+1} - 1, \cdots, n_n)}$$

$$|f_i(n_1, \cdots, n_i + 1, \cdots, n_n)|^2 - p|f_i(n_1, \cdots, n_n)|^2 = 1. \tag{10}$$

Solving the above equations we find

$$f_i(n_1, \cdots, n_n) = q^{\Sigma_{k=i+1}^n n_k} \sqrt{[n_i]}, \tag{11}$$

where $[x]$ is defined as

$$[x] = \frac{p^x - 1}{p - 1}.$$

Thus the representation of this algebra becomes

$$a_i|n_1, \cdots, n_n >= q^{\Sigma_{k=i+1}^n n_k} \sqrt{[n_i]}|n_1, \cdots, n_i - 1, \cdots, n_n >$$

$$a_i^\dagger|n_1, \cdots, n_n >= q^{-\Sigma_{k=i+1}^n n_k} \sqrt{[n_i + 1]}|n_1, \cdots, n_i + 1, \cdots, n_n > . \tag{12}$$

The general eigenstates $|n_1, n_2, \cdots, n_n >$ is obtained by applying $a_i^\dagger$'s operators to the ground state $|0, 0, \cdots, 0>$:

$$|n_1, n_2, \cdots, n_n >= \frac{(a_n^\dagger)^{n_n} \cdots (a_1^\dagger)^{n_1}}{\sqrt{[n_n]! \cdots [n_1]!}} |0, 0, \cdots, 0>, \tag{13}$$

where

$$[n]! = [n][n-1] \cdots [2][1], \quad [0]! = 1.$$

The coherent states for $gl_q(n)$ algebra is usually defined as

$$a_i |z_1, \cdots, z_i, \cdots, z_n >_- = z_i |z_1, \cdots, z_i, \cdots, z_n >_- . \tag{14}$$

From the $gl_q(n)$-covariant oscillator algebra we obtain the following commutation relation between z_i's and z_i^*'s, where z_i^* is a complex conjugate of z_i;

$$\begin{aligned} z_i z_j &= q z_j z_i, \quad (i < j), \\ z_i^* z_j^* &= \frac{1}{q} z_j^* z_i, \quad (i < j), \\ z_i^* z_j &= q z_j z_i^*, \quad (i \neq j) \\ z_i^* z_i &= z_i z_i^*. \end{aligned} \tag{15}$$

Using these relations the coherent state becomes

$$|z_1, \cdots, z_n >_- = c(z_1, \cdots, z_n) \Sigma_{n_1, \cdots, n_n = 0}^{\infty} \frac{z_n^{n_n} \cdots z_1^{n_1}}{\sqrt{[n_1]! \cdots [n_n]!}} |n_1, n_2, \cdots, n_n > . \tag{16}$$

Using the eq.(13) we can rewrite eq.(16) as

$$|z_1,\cdots,z_n>_-=c(z_1,\cdots,z_n)e_p(z_na_n^\dagger)\cdots e_p(z_1a_1^\dagger)|0,0,\cdots,0>, \tag{17}$$

where

$$e_p(x)=\Sigma_{n=0}^{\infty}\frac{x^n}{[n]!}$$

is a deformed exponential function.

In order to obtain the normalized coherent states, we should impose the condition $_<z_1,\cdots,z_n|z_1,\cdots,z_n>_-=1$. Then the normalized coherent states are given by

$$|z_1,\cdots,z_n>_-=\frac{1}{\sqrt{e_p(|z_1|^2)\cdots e_p(|z_n|^2)}}e_p(z_na_n^\dagger)\cdots e_p(z_1a_1^\dagger)|0,0,\cdots,0>, \tag{18}$$

where $|z_i|^2=z_iz_i^*=z_i^*z_i$.

2.2 Positive Energy Coherent States

The purpose of this subsection is to obtain another type of coherent states for algebra (1). In order to do so , it is convenient to introduce n subhamiltonians as follows

$$H_i=a_i^\dagger a_i-\nu,$$

where

$$\nu=\frac{1}{1-p}.$$

Then the commutation relation between the subhamiltonians and mode operators are given by

$$H_i a_j^\dagger = (\delta_{ij}(p-1)+1) a_j^\dagger H_i, \qquad [H_i, H_j] = 0. \tag{19}$$

Acting subhamiltonian on the number eigenstates gives

$$H_i|n_1, n_2, \cdots, n_n >= -\frac{p^{n_i}}{1-p}|n_1, n_2, \cdots, n_n > \tag{20}$$

Thus the energy becomes negative when $0 < p < 1$. As was noticed in ref [13], for the positive energy states it is not a_i but $a_i^\dagger$ that play a role of the lowering operator:

$$H_i|\lambda_1 p^{n_1}, \cdots, \lambda_n p^{n_n} >= \lambda_i p^{n_i}|\lambda_1 p^{n_1}, \cdots, \lambda_n p^{n_n} >$$

$$a_i^\dagger|\lambda_1 p^{n_1}, \cdots, \lambda_n p^{n_n} >= q^{-\Sigma_{k=i+1}^n n_k}\sqrt{\lambda_i p^{n_i+1} + \nu}|\lambda_1 p^{n_1}, \cdots, \lambda_i p^{n_i+1}, \cdots, \lambda_n p^{n_n} >$$

$$a_i|\lambda_1 p^{n_1}, \cdots, \lambda_n p^{n_n} >= q^{\Sigma_{k=i+1}^n n_k}\sqrt{\lambda_i p^{n_i} + \nu}|\lambda_1 p^{n_1}, \cdots, \lambda_i p^{n_i-1}, \cdots, \lambda_n p^{n_n} >, \tag{21}$$

where $\lambda_1, \cdots, \lambda_n > 0$.

Due to this fact, it is natural to define coherent states corresponding to the representation (21) as the eigenstates of $a_i^\dagger$'s:

$$a_i^\dagger|z_1, \cdots, z_n >_+ = z_i|z_1, \cdots, z_n >_+ \tag{22}$$

Because the representation (21) depends on n free paprameters λ_i's , the coherent states $|z_1, \cdots, z_n>_+$ can take different forms.

If we assume that the positive energy states are normalizable, i.e. $<\lambda_1 p^{n_1}, \cdots, \lambda_n p^{n_n}|\lambda_1 p^{n'_1}, \cdots, \lambda_n p^{n'_n}>= \delta_{n_1 n'_1} \cdots \delta_{n_n n'_n}$, and form exactly one series for some fixed λ_i's, then we can obtain

$$|z_1, \cdots, z_n>_+$$

$$= C\Sigma^{\infty}_{n_1,\cdots,n_n=-\infty}\left[\Pi^n_{k=0}\frac{p^{\frac{n_k(n_k-1)}{4}}}{\sqrt{(-\frac{\nu}{\lambda_k};p)_{n_k}}}\left(\frac{1}{\sqrt{\lambda_k}}\right)^{n_k}\right] z_n^{n_n}\cdots z_1^{n_1}|\lambda_1 p^{-n_1}, \cdots, \lambda_n p^{-n_n}> . \tag{23}$$

If we demand that $_+< z_1, \cdots, z_n|z_1, \cdots, z_n>_+= 1$, we have

$$C^{-2} = \Pi^n_{k=1}{}_0\psi_1(-\frac{\nu}{\lambda_k}; p, -\frac{|z_k|^2}{\lambda_k}) \tag{24}$$

where bilateral p-hypergeometric series ${}_0\psi_1(a; p, x)$is defined by [14]

$${}_0\psi_1(a; p, x) = \Sigma^{\infty}_{n=-\infty}\frac{(-)^n p^{n(n-1)/2}}{(a; p)_n}x^n. \tag{25}$$

2.3 Two Parameter Deformed $gl(n)$ Algebra

The purpose of this subsection is to derive the deformed $gl(n)$ algebra from the deformed multimode oscillator algebra. The multimode oscillators given in eq.(1) can be arrayed in bilinears to construct the generators

$$E_{ij} = a^{\dagger}_i a_j. \tag{26}$$

From the fact that $a_i^\dagger$ is a hermitian adjoint of a_i, we know that

$$E_{ij}^\dagger = E_{ji}. \tag{27}$$

Then the deformed $gl(n)$ algebra is obtained from the algebra (1):

$$[E_{ii}, E_{jj}] = 0,$$

$$[E_{ii}, E_{jk}] = 0, \quad (i \neq j \neq k)$$

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj}, \quad (i \neq j)$$

$$E_{ii}E_{ij} - pE_{ij}E_{ii} = E_{ij}, \quad (i \neq j)$$

$$E_{ij}E_{ik} = \begin{cases} q^{-1}E_{ik}E_{ij} & \text{if } j < k \\ qE_{ik}E_{ij} & \text{if } j > k \end{cases}$$

$$E_{ij}E_{kl} = q^{2(R(i,k)+R(j,l)-R(j,k)-R(i,l))}E_{kl}E_{ij}, \quad (i \neq j \neq k \neq l), \tag{28}$$

where the symbol $R(i,j)$ is defined by

$$R(i,j) = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

This algebra goes to an ordinary $gl(n)$ algebra when the deformation parameters q and p goes to 1.

3 q-symmetric states

In this section we study the statistics of many particle state. Let N be the number of particles. Then the N-partcle state can be obtained from the tensor product of single particle state:

$$|i_1, \cdots, i_N >= |i_1 > \otimes |i_2 > \otimes \cdots \otimes |i_N >, \tag{29}$$

where $i_1, \cdots, i_N$ take one value among $\{1, 2, \cdots, n\}$ and the sigle particle state is defined by $|i_k >= a^\dagger_{i_k}|0>$.

Consider the case that k appears n_k times in the set $\{i_1, \cdots, i_N\}$. Then we have

$$n_1 + n_2 + \cdots + n_n = \sum_{k=1}^{n} n_k = N. \tag{30}$$

Using these facts we can define the q-symmetric states as follows:

$$|i_1, \cdots, i_N >_q = \sqrt{\frac{[n_1]_{p^2}! \cdots [n_n]_{p^2}!}{[N]_{p^2}!}} \sum_{\sigma \in Perm} \mathrm{sgn}_q(\sigma) |i_{\sigma(1)} \cdots i_{\sigma(N)} >, \tag{31}$$

where

$$\begin{aligned} \mathrm{sgn}_q(\sigma) &= q^{R(i_1 \cdots i_N)} p^{R(\sigma(1) \cdots \sigma(N))}, \\ R(i_1, \cdots, i_N) &= \sum_{k=1}^{N} \sum_{l=k+1}^{N} R(i_k, i_l) \end{aligned} \tag{32}$$

and $[x]_{p^2} = \frac{p^{2x}-1}{p^2-1}$. Then the q-symmetric states obeys

$$| \cdots, i_k, i_{k+1}, \cdots >_q = \begin{cases} q^{-1} | \cdots, i_{k+1}, i_k, \cdots >_q & \text{if } i_k < i_{k+1} \\ | \cdots, i_{k+1}, i_k, \cdots >_q & \text{if } i_k = i_{k+1} \\ q | \cdots, i_{k+1}, i_k, \cdots >_q & \text{if } i_k > i_{k+1} \end{cases} \tag{33}$$

The above property can be rewritten by introducing the deformed transition operator $P_{k,k+1}$ obeying

$$P_{k,k+1}|\cdots, i_k, i_{k+1}, \cdots >_q = |\cdots, i_{k+1}, i_k, \cdots >_q \tag{34}$$

This operator satisfies

$$P_{k+1,k}P_{k,k+1} = Id, \quad \text{so} \quad P_{k+1,k} = P_{k,k+1}^{-1} \tag{35}$$

Then the equation (33) can be written as

$$P_{k,k+1}|\cdots, i_k, i_{k+1}, \cdots >_q = q^{-\epsilon(i_k, i_{k+1})}|\cdots, i_{k+1}, i_k, \cdots >_q \tag{36}$$

where $\epsilon(i,j)$ is defined as

$$\epsilon(i,j) = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j \end{cases}$$

It is worth noting that the relation (36) does not contain the deformation parameter p. And the relation (36) goes to the symmetric relation for the ordinary bosons when the deformation parameter q goes to 1. If we define the fundamental q-symmetric state $|q>$ as

$$|q> = |i_1, i_2, \cdots, i_N >_q$$

with $i_1 \leq i_2 \leq \cdots \leq i_N$, we have for any k

$$|P_{k,k+1}|q>|^2 = ||q>|^2 = 1.$$

In deriving the above relation we used following identity

$$\sum_{\sigma\in Perm} p^{R(\sigma(1),\cdots,\sigma(N))} = \frac{[N]_{p^2}!}{[n_1]_{p^2}!\cdots[n_n]_{p^2}!}.$$

4 Concluding Remark

To conclude, I used the two parameter deformed multimode oscillator system given in ref [12] to construct its representation, coherent states and deformed $gl_q(n)$ algebra. Mutimode oscillator is important when we investigate the many body quantum mechanics and statistical mechanics. In order to construct the new statistical behavior for deformed particle obeying the algebra (1), I investigated the defomed symmetric property of two parameter deformed mutimode states.

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